

Contents

Part I Regular Continued Fractions

1	Classical Notions and Definitions	3
1.1	Continued Fractions	3
1.1.1	Definition of a Continued Fraction	3
1.1.2	Regular Continued Fractions for Rational Numbers	4
1.1.3	Regular Continued Fractions and the Euclidean Algorithm	5
1.1.4	Continued Fractions with Arbitrary Elements	6
1.2	Convergence of Infinite Regular Continued Fractions	6
1.3	Existence and Uniqueness of a Regular Continued Fraction for a Given Real Number	10
1.4	Monotone Behavior of Convergents	12
1.5	Approximation Rates of Regular Continued Fractions	14
1.6	Exercises	18
2	On Integer Geometry	19
2.1	Basic Notions and Definitions	20
2.1.1	Objects and Congruence Relation of Integer Geometry	20
2.1.2	Invariants of Integer Geometry	20
2.1.3	Index of Sublattices	21
2.1.4	Integer Length of Integer Segments	22
2.1.5	Integer Distance to Integer Lines	23
2.1.6	Integer Area of Integer Triangles	24
2.1.7	Index of Rational Angles	24
2.2	Empty Triangles: Their Integer and Euclidean Areas	25
2.3	Integer Area of Polygons	26
2.4	Pick's Formula	29
2.5	The Twelve-Point Theorem	30
2.6	Exercises	30
3	Geometry of Regular Continued Fractions	33
3.1	Classical Construction	33

3.2	Geometric Interpretation of the Elements of Continued Fractions . . .	37
3.3	Index of an Angle, Duality of Sails	38
3.4	Exercises	39
4	Complete Invariant of Integer Angles	41
4.1	Integer Sines of Rational Angles	41
4.2	Sails for Arbitrary Angles and Their LLS Sequences	43
4.3	On Complete Invariants of Angles with Integer Vertex	43
4.4	Exercises	46
5	Integer Trigonometry for Integer Angles	47
5.1	Definition of Trigonometric Functions	47
5.2	Basic Properties of Integer Trigonometry	48
5.3	Transpose Integer Angles	49
5.4	Adjacent Integer Angles	51
5.5	Right Integer Angles	54
5.6	Opposite Interior Angles	55
5.7	Exercises	55
6	Integer Angles of Integer Triangles	57
6.1	Integer Sine Formula	57
6.2	On Integer Congruence Criteria for Triangles	58
6.3	On Sums of Angles in Triangles	59
6.4	Angles and Segments of Integer Triangles	61
6.5	Examples of Integer Triangles	62
6.6	Exercises	65
7	Continued Fractions and $SL(2, \mathbb{Z})$ Conjugacy Classes. Elements of Gauss's Reduction Theory. Markov Spectrum	67
7.1	Geometric Continued Fractions	67
7.1.1	Definition of a Geometric Continued Fraction	67
7.1.2	Geometric Continued Fractions of Real Spectrum $SL(2, \mathbb{R})$ Matrices	68
7.1.3	Duality of Sails	69
7.1.4	LLS Sequences for Real Spectrum Matrices	69
7.1.5	Algebraic Sails	70
7.1.6	LLS Periods of Real Spectrum Matrices	70
7.2	Geometry of Gauss's Reduction Theory	71
7.2.1	Cases of Matrices with Complex, Real, and Coinciding Eigenvalues	71
7.2.2	Reduced Matrices	72
7.2.3	Reduced Matrices and Integer Conjugacy Classes	73
7.2.4	Complete Invariant of Integer Conjugacy Classes	73
7.2.5	Algebraicity of Matrices with Periodic LLS Sequences	74
7.3	Some Technical Details and Open Questions Related to Gauss's Reduction Theory	75
7.3.1	Proof of Theorem 7.14	75

7.3.2	Calculation of Periods of the LLS Sequence	77
7.3.3	Complexity of Reduced Operators	78
7.3.4	Frequencies of Reduced Matrices	79
7.4	Minima of Quadratic Forms and the Markov Spectrum	81
7.4.1	Calculation of Minima of Quadratic Forms	81
7.4.2	Some Properties of Markov Spectrum	82
7.4.3	Markov Numbers	83
7.5	Exercises	85
8	Lagrange’s Theorem	87
8.1	The Dirichlet Group	87
8.2	Construction of the Integer n th Root of a $GL(2, \mathbb{Z})$ Matrix	90
8.3	Pell’s Equation	91
8.4	Periodic Continued Fractions and Quadratic Irrationalities	93
8.5	Exercises	96
9	Gauss–Kuzmin Statistics	99
9.1	Some Information from Ergodic Theory	100
9.2	The Measure Space Related to Continued Fractions	101
9.2.1	Definition of the Measure Space Related to Continued Fractions	101
9.2.2	Theorems on Density Points of Measurable Subsets	101
9.3	On the Gauss Map	102
9.3.1	The Gauss Map and Corresponding Invariant Measure	102
9.3.2	An Example of an Invariant Set for the Gauss Map	104
9.3.3	Ergodicity of the Gauss Map	105
9.4	Pointwise Gauss–Kuzmin Theorem	107
9.5	Original Gauss–Kuzmin Theorem	108
9.6	Cross-Ratio in Projective Geometry	108
9.6.1	Projective Linear Group	109
9.6.2	Cross-Ratio, Infinitesimal Cross-Ratio	109
9.7	Smooth Manifold of Geometric Continued Fractions	110
9.8	Möbius Measure on the Manifolds of Continued Fractions	110
9.9	Explicit Formulas for the Möbius Form	111
9.10	Relative Frequencies of Edges of One-Dimensional Continued Fractions	112
9.11	Exercises	114
10	Geometric Aspects of Approximation	115
10.1	Two Types of Best Approximations of Rational Numbers	115
10.1.1	Best Diophantine Approximations	115
10.1.2	Strong Best Diophantine Approximations	120
10.2	Rational Approximations of Arrangements of Two Lines	122
10.2.1	Regular Angles and Related Markov–Davenport Forms	123
10.2.2	Integer Arrangements and Their Sizes	124
10.2.3	Discrepancy Functional and Approximation Model	124

10.2.4	Lagrange Estimates for the Case of Continued Fractions with Bounded Elements	125
10.2.5	Periodic Sails and Best Approximations in the Algebraic Case	130
10.2.6	Finding Best Approximations of Line Arrangements	132
10.3	Exercises	135
11	Geometry of Continued Fractions with Real Elements and Kepler's Second Law	137
11.1	Continued Fractions with Integer Coefficients	137
11.2	Continued Fractions with Real Coefficients	139
11.2.1	Broken Lines Related to Sequences of Arbitrary Real Numbers	140
11.2.2	Continued Fractions Related to Broken Lines	142
11.2.3	Geometry of Continued Fractions for Broken Lines	143
11.3	Areal and Angular Densities for Differentiable Curves	146
11.3.1	Notions of Real and Angular Densities	146
11.3.2	Curves and Broken Lines	148
11.3.3	Some Examples	149
11.4	Exercises	151
12	Extended Integer Angles and Their Summation	153
12.1	Extension of Integer Angles. Notion of Sums of Integer Angles	153
12.1.1	Extended Integer Angles and Revolution Number	154
12.1.2	On Normal Forms of Extended Angles	157
12.1.3	Trigonometry of Extended Angles. Associated Integer Angles	161
12.1.4	Opposite Extended Angles	162
12.1.5	Sums of Extended Angles	162
12.1.6	Sums of Integer Angles	164
12.2	Relations Between Extended and Integer Angles	164
12.3	Proof of Theorem 6.8(i)	165
12.3.1	Two Preliminary Lemmas	166
12.3.2	Conclusion of the Proof of Theorem 6.8(i)	170
12.4	Exercises	172
13	Integer Angles of Polygons and Global Relations for Toric Singularities	173
13.1	Theorem on Angles of Integer Convex Polygons	173
13.2	Toric Surfaces and Their Singularities	175
13.2.1	Definition of Toric Surfaces	175
13.2.2	Singularities of Toric Surfaces	176
13.3	Relations on Toric Singularities of Surfaces	178
13.3.1	Toric Singularities of n -gons with Fixed Parameter n	178
13.3.2	Realizability of a Prescribed Set of Toric Singularities	179
13.4	Exercises	182

Part II Multidimensional Continued Fractions

14 Basic Notions and Definitions of Multidimensional Integer Geometry	185
14.1 Basic Integer Invariants in Integer Geometry	185
14.1.1 Objects and the Congruence Relation	185
14.1.2 Integer Invariants and Indices of Sublattices	186
14.1.3 Integer Volume of Simplices	187
14.1.4 Integer Angle Between Two Planes	187
14.1.5 Integer Distance Between Two Disjoint Planes	188
14.2 Integer and Euclidean Volumes of Basis Simplices	189
14.3 Integer Volumes of Polyhedra	191
14.3.1 Interpretation of Integer Volumes of Simplices via Euclidean Volumes	192
14.3.2 Integer Volume of Polyhedra	192
14.3.3 Decomposition into Empty Simplices	193
14.4 Lattice Plücker Coordinates and Calculation of Integer Volumes of Simplices	194
14.4.1 Grassmann Algebra on \mathbb{R}^n and k -Forms	194
14.4.2 Plücker Coordinates	195
14.4.3 Oriented Lattices in \mathbb{R}^n and Their Lattice Plücker Embedding	196
14.4.4 Lattice Plücker Coordinates and Integer Volumes of Simplices	197
14.5 Ehrhart Polynomials as Generalized Pick's Formula	198
14.6 Exercises	200
15 On Empty Simplices, Pyramids, Parallelepipeds	203
15.1 Classification of Empty Integer Tetrahedra	203
15.2 Classification of Completely Empty Lattice Pyramids	204
15.3 Two Open Problems Related to the Notion of Emptiness	206
15.3.1 Problem on Empty Simplices	206
15.3.2 Lonely Runner Conjecture	207
15.4 Proof of White's Theorem and the Empty Tetrahedra Classification Theorems	208
15.4.1 IDC-System	208
15.4.2 A Lemma on Sections of an Integer Parallelepiped	209
15.4.3 A Corollary on Integer Distances Between the Vertices and the Opposite Faces of a Tetrahedron with Empty Faces	209
15.4.4 Lemma on One Integer Node	210
15.4.5 Proof of White's Theorem	211
15.4.6 Deduction of Corollary 15.3 from White's Theorem	213
15.5 Exercises	214
16 Multidimensional Continued Fractions in the Sense of Klein	215
16.1 Background	215
16.2 Some Notation and Definitions	216
16.2.1 A-Hulls and Their Boundaries	216

16.2.2	Definition of Multidimensional Continued Fraction in the Sense of Klein	217
16.2.3	Face Structure of Sails	217
16.3	Finite Continued Fractions	218
16.4	On a Generalized Kronecker's Approximation Theorem	219
16.4.1	Addition of Sets in \mathbb{R}^n	219
16.4.2	Integer Approximation Spaces and Affine Irrational Vectors	220
16.4.3	Formulation of the Theorem	221
16.4.4	Proof of the Multidimensional Kronecker's Approximation Theorem	221
16.5	Polyhedral Structure of Sails	224
16.5.1	The Intersection of the Closures of A-Hulls with Faces of Corresponding Cones	224
16.5.2	Homeomorphic Types of Sails	226
16.5.3	Combinatorial Structure of Sails for Cones in General Position	228
16.5.4	A-Hulls and Quasipolyhedra	231
16.6	Two-Dimensional Faces of Sails	232
16.6.1	Faces with Integer Distance to the Origin Equal One	232
16.6.2	Faces with Integer Distance to the Origin Greater than One	234
16.7	Exercises	234
17	Dirichlet Groups and Lattice Reduction	237
17.1	Orders, Units, and Dirichlet's Unit Theorem	237
17.2	Dirichlet Groups and Groups of Units in Orders	238
17.2.1	Notion of a Dirichlet Group	238
17.2.2	On Isomorphisms of Dirichlet Groups and Certain Groups of Units	239
17.2.3	Dirichlet Groups Related to Orders That Do not Have Complex Roots of Unity	240
17.3	Calculation of a Basis of the Additive Group $\Gamma(A)$	241
17.3.1	Step 1: Preliminary Statements	241
17.3.2	Step 2: Application of the LLL-Algorithm	242
17.3.3	Step 3: Calculation of an Integer Basis Having a Basis of an Integer Sublattice	242
17.4	Calculation of a Basis of the Positive Dirichlet Group $\mathcal{E}_+(A)$	243
17.5	Lattice Reduction and the LLL-Algorithm	243
17.5.1	Reduced Bases	244
17.5.2	The LLL-Algorithm	245
17.6	Exercises	246
18	Periodicity of Klein Polyhedra. Generalization of Lagrange's Theorem	249
18.1	Continued Fractions Associated to Matrices	249
18.2	Algebraic Periodic Multidimensional Continued Fractions	250

18.3	Torus Decompositions of Periodic Sails in \mathbb{R}^3	251
18.4	Three Single Examples of Torus Decompositions in \mathbb{R}^3	253
18.5	Examples of Infinite Series of Torus Decomposition	257
18.6	Two-Dimensional Continued Fractions Associated to Transpose Frobenius Normal Forms	261
18.7	Some Problems and Conjectures on Periodic Geometry of Algebraic Sails	262
18.8	Generalized Lagrange's Theorem	265
18.9	Littlewood and Oppenheim Conjectures in the Framework of Multidimensional Continued Fractions	269
18.10	Exercises	270
19	Multidimensional Gauss–Kuzmin Statistics	271
19.1	Möbius Measure on the Manifold of Continued Fractions	271
19.1.1	Smooth Manifold of n -Dimensional Continued Fractions	271
19.1.2	Möbius Measure on the Manifolds of Continued Fractions	272
19.2	Explicit Formulae for the Möbius Form	273
19.3	Relative Frequencies of Faces of Multidimensional Continued Fractions	275
19.4	Some Calculations of Frequencies for Faces in the Two-Dimensional Case	276
19.4.1	Some Hints for Computation of Approximate Values of Relative Frequencies	276
19.4.2	Numeric Calculations of Relative Frequencies	277
19.4.3	Two Particular Results on Relative Frequencies	279
19.5	Exercises	279
20	On Construction of Multidimensional Continued Fractions	281
20.1	Inductive Algorithm	281
20.1.1	Some Background	281
20.1.2	Description of the Algorithm	282
20.1.3	Step 1a: Construction of the First Hyperface	283
20.1.4	Step 1b, 4: How Decompose the Polytope into Its Faces	284
20.1.5	Step 2: Construction of the Adjacent Hyperface	284
20.1.6	Step 2: Test of the Equivalence Class for the Hyperface F' to Have Representatives in the Set of Hyperfaces D	285
20.2	Deductive Algorithms to Construct Sails	285
20.2.1	General Idea of Deductive Algorithms	285
20.2.2	The First Deductive Algorithm	286
20.2.3	The Second Deductive Algorithm	287
20.2.4	Test of the Conjectures Produced in the Two-Dimensional Case	290
20.2.5	On the Verification of a Conjecture for the Multidimensional Case	296
20.3	An Example of the Calculation of a Fundamental Domain	297
20.4	Exercise	300

21	Gauss Reduction in Higher Dimensions	301
21.1	Organization of This Chapter	301
21.2	Hessenberg Matrices and Conjugacy Classes	302
21.2.1	Notions and Definitions	303
21.2.2	Construction of Perfect Hessenberg Matrices Conjugate to a Given One	305
21.2.3	Existence and Finiteness of ζ -Reduced Hessenberg Matrices	307
21.2.4	Families of Hessenberg Matrices with Given Hessenberg Type	308
21.2.5	ζ -Reduced Matrices in the 2-Dimensional Case	311
21.3	Complete Geometric Invariant of Conjugacy Classes	312
21.3.1	Continued Fractions in the Sense of Klein–Voronoi	312
21.3.2	Geometric Complete Invariants of Dirichlet Groups	316
21.3.3	Geometric Invariants of Conjugacy Classes	317
21.4	Algorithmic Aspects of Reduction to ζ -Reduced Matrices	318
21.4.1	Markov–Davenport Characteristics	318
21.4.2	Klein–Voronoi Continued Fractions and Minima of MD-Characteristics	321
21.4.3	Construction of ζ -Reduced Matrices by Klein–Voronoi Continued Fractions	322
21.5	Diophantine Equations Related to the Markov–Davenport Characteristic	324
21.5.1	Multidimensional w -Sails and w -Continued Fractions	324
21.5.2	Solution of Eq. (21.1)	326
21.6	On Reduced Matrices in $SL(3, \mathbb{Z})$ with Two Complex Conjugate Eigenvalues	327
21.6.1	Perfect Hessenberg Matrices of a Given Hessenberg Type	327
21.6.2	Parabolic Structure of the Set of NRS-Matrices	328
21.6.3	Theorem on Asymptotic Uniqueness of ζ -Reduced NRS-Matrices	329
21.6.4	Examples of NRS-Matrices for a Given Hessenberg Type	331
21.6.5	Proof of Theorem 21.43	333
21.6.6	Proof of Theorem 21.48	336
21.7	Open Problems	342
21.8	Exercises	345
22	Approximation of Maximal Commutative Subgroups	347
22.1	Rational Approximations of MCRS-Groups	347
22.1.1	Maximal Commutative Subgroups and Corresponding Simplicial Cones	348
22.1.2	Regular Subgroups and Markov–Davenport Forms	349
22.1.3	Rational Subgroups and Their Sizes	350
22.1.4	Discrepancy Functional	351
22.1.5	Approximation Model	351

22.1.6	Diophantine Approximation and MCRS-Group Approximation	352
22.2	Simultaneous Approximation in \mathbb{R}^3 and MCRS-Group Approximation	353
22.2.1	General Construction	353
22.2.2	A Ray of a Nonreal Spectrum Operator	354
22.2.3	Two-Dimensional Golden Ratio	355
22.3	Exercises	356
23	Other Generalizations of Continued Fractions	357
23.1	Relative Minima	357
23.1.1	Relative Minima and the Minkowski–Voronoi Complex	358
23.1.2	Minkowski–Voronoi Tessellations of the Plane	360
23.1.3	Minkowski–Voronoi Continued Fractions in \mathbb{R}^3	361
23.1.4	Combinatorial Properties of the Minkowski–Voronoi Tessellation for Integer Sublattices	362
23.2	Farey Addition, Farey Tessellation, Triangle Sequences	364
23.2.1	Farey Addition of Rational Numbers	364
23.2.2	Farey Tessellation	365
23.2.3	Descent Toward the Absolute	366
23.2.4	Triangle Sequences	368
23.3	Decompositions of Coordinate Rectangular Bricks and O’Hara’s Algorithm	373
23.3.1	Π -Congruence of Coordinate Rectangular Bricks	374
23.3.2	Criterion of Π -Congruence of Coordinate Bricks	375
23.3.3	Geometric Version of O’Hara’s Algorithm for Partitions	375
23.4	Algorithmic Generalized Continued Fractions	378
23.4.1	General Algorithmic Scheme	378
23.4.2	Examples of Algorithms	379
23.4.3	Algebraic Periodicity	381
23.4.4	A Few Words About Convergents	381
23.5	Branching Continued Fractions	382
23.6	Continued Fractions and Rational Knots and Links	386
23.6.1	Necessary Definitions	386
23.6.2	Rational Tangles and Operations on Them	387
23.6.3	Main Results on Rational Knots and Tangles	388
	References	391
	Index	401