

# Contents

|          |                                                                           |           |
|----------|---------------------------------------------------------------------------|-----------|
| <b>1</b> | <b>Partial Differential Equations and Their Classification Into Types</b> | <b>1</b>  |
| 1.1      | Examples                                                                  | 1         |
| 1.2      | Classification of Second-Order Equations into Types                       | 5         |
| 1.3      | Type Classification for Systems of First Order                            | 7         |
| 1.4      | Characteristic Properties of the Different Types                          | 9         |
| 1.5      | Literature                                                                | 12        |
| <b>2</b> | <b>The Potential Equation</b>                                             | <b>13</b> |
| 2.1      | Posing the Problem                                                        | 13        |
| 2.2      | Singularity Function                                                      | 16        |
| 2.3      | Mean-Value Property and Maximum Principle                                 | 19        |
| 2.4      | Continuous Dependence on the Boundary Data                                | 25        |
| <b>3</b> | <b>The Poisson Equation</b>                                               | <b>29</b> |
| 3.1      | Posing the Problem                                                        | 29        |
| 3.2      | Representation of the Solution by the Green Function                      | 30        |
| 3.3      | Existence of a Solution                                                   | 33        |
| 3.4      | The Green Function for the Ball                                           | 38        |
| 3.5      | The Neumann Boundary-Value Problem                                        | 39        |
| 3.6      | The Integral Equation Method                                              | 41        |
| <b>4</b> | <b>Difference Methods for the Poisson Equation</b>                        | <b>43</b> |
| 4.1      | Introduction: The One-Dimensional Case                                    | 44        |
| 4.2      | The Five-Point Formula                                                    | 46        |
| 4.3      | M-matrices, Matrix Norms, Positive-Definite Matrices                      | 50        |
| 4.4      | Properties of the Matrix $L_h$                                            | 59        |
| 4.5      | Convergence                                                               | 66        |
| 4.6      | Discretisations of Higher Order                                           | 69        |

|          |                                                                                    |            |
|----------|------------------------------------------------------------------------------------|------------|
| 4.7      | The Discretisation of the Neumann Boundary-Value Problem . . . . .                 | 72         |
| 4.7.1    | One-Sided Difference for $\partial u / \partial n$ . . . . .                       | 73         |
| 4.7.2    | Symmetric Difference for $\partial u / \partial n$ . . . . .                       | 77         |
| 4.7.3    | Symmetric Difference for $\partial u / \partial n$ on an Offset Grid . . . . .     | 79         |
| 4.7.4    | Proof of the Stability Theorem 4.62 . . . . .                                      | 79         |
| 4.8      | Discretisation in an Arbitrary Domain . . . . .                                    | 86         |
| 4.8.1    | Shortley–Weller Approximation . . . . .                                            | 86         |
| 4.8.2    | Interpolation in Near-Boundary Points . . . . .                                    | 90         |
| <b>5</b> | <b>General Boundary-Value Problems</b> . . . . .                                   | <b>93</b>  |
| 5.1      | Dirichlet Boundary-Value Problems for Linear Differential Equations . . . . .      | 93         |
| 5.1.1    | Posing the Problem . . . . .                                                       | 93         |
| 5.1.2    | Maximum Principle . . . . .                                                        | 95         |
| 5.1.3    | Uniqueness of the Solution and Continuous Dependence . . . . .                     | 98         |
| 5.1.4    | Difference Methods for the General Differential Equation of Second Order . . . . . | 100        |
| 5.1.5    | Green's Function . . . . .                                                         | 105        |
| 5.2      | General Boundary Conditions . . . . .                                              | 106        |
| 5.2.1    | Formulating the Boundary-Value Problem . . . . .                                   | 106        |
| 5.2.2    | Difference Methods for General Boundary Conditions . . . . .                       | 109        |
| 5.3      | Boundary Problems of Higher Order . . . . .                                        | 113        |
| 5.3.1    | The Biharmonic Differential Equation . . . . .                                     | 113        |
| 5.3.2    | General Linear Differential Equations of Order $2m$ . . . . .                      | 113        |
| 5.3.3    | Discretisation of the Biharmonic Differential Equation . . . . .                   | 115        |
| <b>6</b> | <b>Tools from Functional Analysis</b> . . . . .                                    | <b>119</b> |
| 6.1      | Banach Spaces and Hilbert Spaces . . . . .                                         | 119        |
| 6.1.1    | Normed Spaces . . . . .                                                            | 119        |
| 6.1.2    | Operators . . . . .                                                                | 120        |
| 6.1.3    | Banach Spaces . . . . .                                                            | 121        |
| 6.1.4    | Hilbert Spaces . . . . .                                                           | 123        |
| 6.2      | Sobolev Spaces . . . . .                                                           | 125        |
| 6.2.1    | $L^2(\Omega)$ . . . . .                                                            | 125        |
| 6.2.2    | $H^k(\Omega)$ and $H_0^k(\Omega)$ . . . . .                                        | 127        |
| 6.2.3    | Fourier Transformation and $H^k(\mathbb{R}^n)$ . . . . .                           | 130        |
| 6.2.4    | $H^s(\Omega)$ for Real $s \geq 0$ . . . . .                                        | 133        |
| 6.2.5    | Trace and Extension Theorems . . . . .                                             | 134        |
| 6.3      | Dual Spaces . . . . .                                                              | 142        |
| 6.3.1    | Dual Space of a Normed Space . . . . .                                             | 142        |
| 6.3.2    | Adjoint Operators . . . . .                                                        | 143        |
| 6.3.3    | Scales of Hilbert Spaces . . . . .                                                 | 145        |
| 6.4      | Compact Operators . . . . .                                                        | 147        |
| 6.5      | Bilinear Forms . . . . .                                                           | 151        |

|          |                                                          |     |
|----------|----------------------------------------------------------|-----|
| <b>7</b> | <b>Variational Formulation</b>                           | 159 |
| 7.1      | Historical Remarks About the Dirichlet Principle         | 159 |
| 7.2      | Equations with Homogeneous Dirichlet Boundary Conditions | 161 |
| 7.2.1    | Dirichlet Boundary Condition                             | 161 |
| 7.2.2    | Weak Formulation                                         | 162 |
| 7.2.3    | $H_0^m(\Omega)$ -Ellipticity                             | 164 |
| 7.2.4    | $H_0^m(\Omega)$ -Coercivity                              | 167 |
| 7.3      | Inhomogeneous Dirichlet Boundary Conditions              | 168 |
| 7.4      | Natural Boundary Conditions                              | 170 |
| 7.4.1    | Variation in $H^m(\Omega)$                               | 170 |
| 7.4.2    | Conormal Boundary Condition                              | 171 |
| 7.4.3    | Oblique Boundary Condition                               | 173 |
| 7.4.4    | Boundary Conditions for $m \geq 2$                       | 176 |
| 7.4.5    | Further Boundary Conditions                              | 178 |
| 7.5      | Pseudo-Differential Equations                            | 179 |
| <b>8</b> | <b>The Finite-Element Method</b>                         | 181 |
| 8.1      | Historical Remarks                                       | 181 |
| 8.2      | The Ritz–Galerkin Method                                 | 183 |
| 8.2.1    | Basics                                                   | 183 |
| 8.2.2    | Analysis of the Discrete Equation                        | 186 |
| 8.2.3    | Solvability of the Discrete Problem                      | 190 |
| 8.2.4    | Examples                                                 | 192 |
| 8.3      | Error Estimates                                          | 194 |
| 8.3.1    | Quasi-Optimality                                         | 194 |
| 8.3.2    | Convergence of the Ritz–Galerkin Solutions               | 195 |
| 8.3.3    | Ritz Projection                                          | 197 |
| 8.3.4    | Further Stability and Error Estimates                    | 198 |
| 8.4      | Finite Elements                                          | 200 |
| 8.4.1    | Introduction: Linear Elements for $\Omega = (a, b)$      | 200 |
| 8.4.2    | Linear Elements for $\Omega \subset \mathbb{R}^2$        | 203 |
| 8.4.3    | Bilinear Elements for $\Omega \subset \mathbb{R}^2$      | 206 |
| 8.4.4    | Quadratic Elements for $\Omega \subset \mathbb{R}^2$     | 208 |
| 8.4.5    | Elements for $\Omega \subset \mathbb{R}^3$               | 209 |
| 8.4.6    | Handling of Side Conditions                              | 210 |
| 8.5      | Error Estimates for Finite-Element Methods               | 213 |
| 8.5.1    | Preparations                                             | 213 |
| 8.5.2    | Properties of Sequences of Finite-Element Spaces         | 216 |
| 8.5.3    | $H^1$ -Estimates for Linear Elements                     | 218 |
| 8.5.4    | $L^2$ Estimates for Linear Elements                      | 220 |
| 8.6      | Generalisations                                          | 224 |
| 8.6.1    | Error Estimates for Other Elements                       | 224 |
| 8.6.2    | Finite Elements for Equations of Higher Order            | 225 |
| 8.6.3    | Finite Elements for Non-Polygonal Regions                | 227 |
| 8.7      | A-posteriori Error Estimates, Adaptivity                 | 230 |

|           |                                                                       |            |
|-----------|-----------------------------------------------------------------------|------------|
| 8.7.1     | A-posteriori Error Estimates                                          | 230        |
| 8.7.2     | Efficiency of the Finite-Element Method                               | 236        |
| 8.7.3     | Adaptive Finite-Element Method                                        | 237        |
| 8.8       | Properties of the System Matrix                                       | 241        |
| 8.8.1     | Connection of $\mathbf{L}$ and $L_h$                                  | 241        |
| 8.8.2     | Equivalent Norms and Mass Matrix                                      | 241        |
| 8.8.3     | Inverse Estimate and Condition of $\mathbf{L}$                        | 244        |
| 8.8.4     | Element Matrices                                                      | 246        |
| 8.8.5     | Positivity, Maximum Principle                                         | 247        |
| 8.9       | Further Remarks                                                       | 248        |
| 8.9.1     | Mixed and Hybrid Finite-Element Methods                               | 248        |
| 8.9.2     | Nonconforming Elements                                                | 249        |
| 8.9.3     | Inadmissible Triangulations                                           | 251        |
| 8.9.4     | Trefftz' Method                                                       | 252        |
| 8.9.5     | Finite-Element Methods for Singular Solutions                         | 253        |
| 8.9.6     | Hierarchical Bases                                                    | 253        |
| 8.9.7     | Superconvergence                                                      | 254        |
| 8.9.8     | Mortar Finite Elements                                                | 255        |
| 8.9.9     | Composite Finite Elements                                             | 257        |
| 8.9.10    | Related Discretisations                                               | 258        |
| 8.9.11    | Sparse Grids                                                          | 260        |
| <b>9</b>  | <b>Regularity</b>                                                     | <b>263</b> |
| 9.1       | Solutions of the Boundary-Value Problem in $H^s(\Omega)$ , $s > m$    | 263        |
| 9.1.1     | The Regularity Problem                                                | 264        |
| 9.1.2     | Regularity Theorems for $\Omega = \mathbb{R}^n$                       | 266        |
| 9.1.3     | Regularity Theorems for $\Omega = \mathbb{R}_+^n$                     | 274        |
| 9.1.4     | Regularity Theorems for General Domains $\Omega \subset \mathbb{R}^n$ | 278        |
| 9.1.5     | Regularity for Convex Domains and Domains with Corners                | 282        |
| 9.2       | Regularity in the Interior                                            | 285        |
| 9.2.1     | Estimates                                                             | 286        |
| 9.2.2     | Behaviour of the Singularity and Green's Function                     | 286        |
| 9.3       | Regularity Properties of Difference Equations                         | 290        |
| 9.3.1     | Discrete $H^1$ -Regularity                                            | 290        |
| 9.3.2     | Consistency                                                           | 296        |
| 9.3.3     | Optimal Error Estimates                                               | 303        |
| 9.3.4     | $H_{0,h}^{m+\theta}$ -Regularity for $-1/2 < \theta < 1/2$            | 305        |
| 9.3.5     | $H_h^2$ -Regularity                                                   | 306        |
| 9.3.6     | Interior Regularity                                                   | 309        |
| <b>10</b> | <b>Special Differential Equations</b>                                 | <b>311</b> |
| 10.1      | Differential Equations with Discontinuous Coefficients                | 311        |
| 10.1.1    | Formulation                                                           | 311        |
| 10.1.2    | Finite-Element Discretisation                                         | 314        |
| 10.1.3    | Discretisation by Difference Schemes                                  | 315        |

|           |                                                                    |            |
|-----------|--------------------------------------------------------------------|------------|
| 10.1.4    | Discontinuous Coefficients of the First and Zeroth Derivatives     | 315        |
| 10.2      | A Singular Perturbation Problem                                    | 316        |
| 10.2.1    | The Convection-Diffusion Equation                                  | 316        |
| 10.2.2    | Stable Difference Schemes                                          | 318        |
| 10.2.3    | Finite Elements                                                    | 321        |
| <b>11</b> | <b>Elliptic Eigenvalue Problems</b>                                | <b>329</b> |
| 11.1      | Formulation of Eigenvalue Problems                                 | 329        |
| 11.2      | Finite-Element Discretisation                                      | 331        |
| 11.2.1    | Discretisation                                                     | 331        |
| 11.2.2    | Qualitative Convergence Results                                    | 333        |
| 11.2.3    | Quantitative Convergence Results                                   | 338        |
| 11.2.4    | Consistent Problems                                                | 343        |
| 11.3      | Discretisation by Difference Methods                               | 346        |
| 11.4      | Further Remarks                                                    | 354        |
| <b>12</b> | <b>Stokes Equations</b>                                            | <b>355</b> |
| 12.1      | Elliptic Systems of Differential Equations                         | 355        |
| 12.2      | Variational Formulation                                            | 359        |
| 12.2.1    | Weak Formulation of the Stokes Equations                           | 359        |
| 12.2.2    | Saddle-Point Problems                                              | 360        |
| 12.2.3    | Existence and Uniqueness of the Solution of a Saddle-Point Problem | 363        |
| 12.2.4    | Solvability and Regularity of the Stokes Problem                   | 366        |
| 12.2.5    | A $V_0$ -elliptic Variational Formulation of the Stokes Problem    | 370        |
| 12.3      | Finite-Element Method for the Stokes Problem                       | 371        |
| 12.3.1    | Finite-Element Discretisation of a Saddle-Point Problem            | 371        |
| 12.3.2    | Stability Conditions                                               | 373        |
| 12.3.3    | Stable Finite-Element Spaces for the Stokes Problem                | 374        |
| 12.3.4    | Divergence-Free Elements                                           | 380        |
| <b>A</b>  | <b>Solution of the Exercises</b>                                   | <b>381</b> |
|           | Exercises of Chapter 1                                             | 381        |
|           | Exercises of Chapter 2                                             | 386        |
|           | Exercises of Chapter 3                                             | 391        |
|           | Exercises of Chapter 4                                             | 395        |
|           | Exercises of Chapter 5                                             | 401        |
|           | Exercises of Chapter 6                                             | 404        |
|           | Exercises of Chapter 7                                             | 408        |
|           | Exercises of Chapter 8                                             | 410        |
|           | Exercises of Chapter 9                                             | 413        |
|           | Exercises of Chapter 10                                            | 418        |
|           | Exercises of Chapter 11                                            | 419        |
|           | Exercises of Chapter 12                                            | 425        |

**References** ..... 429

**List of Symbols and Abbreviations** ..... 443

**Index** ..... 449