

Contents

Preface — V

1 The Natural, Integral, and Rational Numbers — 1

- 1.1 Number Theory and Axiomatic Systems — 1
- 1.2 The Natural Numbers and Induction — 1
- 1.3 The Integers $\mathbb{Z}$ — 11
- 1.4 The Rational Numbers $\mathbb{Q}$ — 14
- 1.5 The Absolute Value in $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ — 17

2 Division and Factorization in the Integers — 20

- 2.1 The Fundamental Theorem of Arithmetic — 20
- 2.2 The Division Algorithm and the Greatest Common Divisor — 24
- 2.3 The Euclidean Algorithm — 27
- 2.4 Least Common Multiples — 31
- 2.5 General Greatest Common Divisors and Lowest Common Divisors — 33

3 Modular Arithmetic — 38

- 3.1 The Ring of Integers Modulo n — 38
- 3.2 Units and the Euler φ -Function — 42
- 3.3 RSA Cryptosystem — 45
- 3.4 The Chinese Remainder Theorem — 46
- 3.5 Quadratic Residues — 53

4 Exceptional Numbers — 59

- 4.1 The Fibonacci Numbers — 59
 - 4.1.1 The Golden Rectangle — 67
 - 4.1.2 Squares in Semicircles — 68
 - 4.1.3 Side Length of a Regular 10-Gon — 69
 - 4.1.4 Construction of the Golden Section with Compass and Straightedge — 70
- 4.2 Perfect Numbers and Mersenne Numbers — 71
- 4.3 Fermat Numbers — 78

5 Pythagorean Triples and Sums of Squares — 82

- 5.1 The Pythagorean Theorem — 82
- 5.2 Classification of the Pythagorean Triples — 84
- 5.3 Sum of Squares — 88

6 Polynomials and Unique Factorization — 95

- 6.1 Polynomials over a Ring — 95
- 6.2 Divisibility in Rings — 99

6.3	The Ring of Polynomials over a Field —	100
6.3.1	The Division Algorithm for Polynomials —	101
6.3.2	Zeros of Polynomials —	103
6.4	Horner-Scheme —	108
6.5	The Euclidean Algorithm and Greatest Common Divisor of Polynomials over Fields —	113
6.5.1	The Euclidean Algorithm for $K[x]$ —	114
6.5.2	Unique Factorization of Polynomials in $K[x]$ —	115
6.5.3	General Unique Factorization Domains —	116
6.6	Polynomial Interpolation and the Shamir Secret Sharing Scheme —	117
6.6.1	Secret Sharing —	117
6.6.2	Polynomial Interpolation over a Field —	117
6.6.3	The Shamir Secret Sharing Scheme —	121
7	Field Extensions and Splitting Fields —	124
7.1	Fields, Subfields and Characteristics —	124
7.2	Field Extensions —	125
7.3	Finite and Algebraic Field Extensions —	130
7.3.1	Finite Fields —	133
7.4	Splitting Fields —	134
8	Permutations and Symmetric Polynomials —	139
8.1	Permutations —	139
8.2	Cycle Decomposition of a Permutation —	142
8.2.1	Conjugate Elements in S_n —	145
8.2.2	Marshall Hall's Theorem —	146
8.3	Symmetric Polynomials —	149
8.4	Some Topics in Group Theory —	154
8.4.1	Cosets and Lagrange's Theorem —	154
8.4.2	Normal Subgroups and Factor Groups —	155
8.4.3	Group Isomorphism Theorems —	156
8.4.4	Solvable Groups —	158
8.4.5	Group Actions and the Sylow Theorems —	160
8.4.6	The Fundamental Theorem of Finitely Generated Abelian Groups —	167
9	Real Numbers —	176
9.1	The Real Number System —	176
9.2	Decimal Representation of Real Numbers —	187
9.3	Periodic Decimal Numbers and the Rational Numbers —	190
9.4	The Uncountability of $\mathbb{R}$ —	192
9.5	Continued Fraction Representation of Real Numbers —	193
9.6	Theorem of Dirichlet and Cauchy's Inequality —	195

9.7	The p -adic Numbers —	197
9.7.1	Normed Fields and Cauchy Completions —	197
9.7.2	The p -adic Fields —	198
9.7.3	The p -adic Norm —	201
9.7.4	The Construction of $\mathbb{Q}_p$ —	202
9.7.5	Ostrowski's Theorem —	203
9.7.6	The p -adic Arithmetic and p -adic Expansions —	204
9.7.7	The p -adic Integers —	209
9.7.8	Principal Ideals, Unique Factorization, and Completeness of $\mathbb{Z}_p$ —	210
9.7.9	Hensel's Lemma and Applications —	212
9.7.10	The Non-Isomorphism of the p -adic Fields —	215

10 The Complex Numbers, the Fundamental Theorem of Algebra, and Polynomial Equations — 219

10.1	The Field $\mathbb{C}$ of Complex Numbers —	219
10.2	The Complex Plane —	223
10.2.1	Geometric Interpretation of Complex Operations —	226
10.2.2	Polar Form and Euler's Identity —	227
10.2.3	Other Constructions of $\mathbb{C}$ —	231
10.2.4	The Gaussian Integers —	231
10.3	The Fundamental Theorem of Algebra —	233
10.3.1	First Proof of the Fundamental Theorem of Algebra —	234
10.3.2	Second Proof of the Fundamental Theorem of Algebra —	237
10.4	Solving Polynomial Equations in terms of Radicals —	239
10.5	Galois Theory and the Solvability of Polynomial Equations in terms of Radicals —	250
10.5.1	Automorphism Groups of Field Extensions —	251
10.5.2	Finite Galois Extensions —	252
10.5.3	The Fundamental Theorem of Galois Theory —	253
10.5.4	Field Extensions by Radicals —	261
10.5.5	Solvability by Radicals and Galois Extensions —	266
10.6	Skew Field Extensions of $\mathbb{C}$ and Frobenius's Theorem —	269

11 Quadratic Number Fields and Pell's Equation — 275

11.1	Algebraic Extensions of $\mathbb{Q}$ —	275
11.2	Algebraic and Transcendental Numbers —	276
11.3	Discriminant and Norm —	278
11.4	Algebraic Integers —	283
11.4.1	The Ring of Algebraic Integers —	284
11.5	Integral Bases —	286
11.6	Quadratic Fields and Quadratic Integers —	288

12	Transcendental Numbers and the Numbers e and π — 296
12.1	The Numbers e and π — 296
12.1.1	Calculation e of π — 299
12.2	The Irrationality of e and π — 303
12.3	The Numbers e and π throughout Mathematics — 310
12.3.1	The Normal Distribution — 310
12.3.2	The Gamma Function and Stirling's Approximation — 311
12.3.3	The Wallis Product Formula — 313
12.4	Existence of a Transcendental Number — 318
12.5	The Transcendence of e and π — 321
12.6	An Amazing Property of π and a Connection to Prime Numbers — 330
13	Compass and Straightedge Constructions and the Classical Problems — 336
13.1	Historical Remarks — 336
13.2	Geometric Constructions — 336
13.3	Four Classical Construction Problems — 343
13.3.1	Squaring the Circle (Problem of Anaxagoras 500–428 BC) — 343
13.3.2	The Doubling of the Cube or the Problem from Delos — 344
13.3.3	The Trisection of an Angle — 344
13.3.4	Construction of a Regular n -Gon — 345
14	Euclidean Vector Spaces — 350
14.1	Length and Angle — 350
14.2	Orthogonality and Applications in $\mathbb{R}^2$ and $\mathbb{R}^3$ — 356
14.3	Orthonormalization and Closest Vector — 365
14.4	Polynomial Approximation — 369
14.5	Secret Sharing Scheme using the Closest Vector Theorem — 371
Bibliography — 375	
Index — 377	