

Contents

1	Setting the Scene	1
1.1	What Is a Differential Equation?	1
1.1.1	Concepts	2
1.2	The Solution and Its Properties	4
1.2.1	An Ordinary Differential Equation	4
1.3	A Numerical Method	6
1.4	Cauchy Problems	10
1.4.1	First-Order Homogeneous Equations	11
1.4.2	First-Order Nonhomogeneous Equations	14
1.4.3	The Wave Equation	15
1.4.4	The Heat Equation	18
1.5	Exercises	20
1.6	Projects	28
2	Two-Point Boundary Value Problems	39
2.1	Poisson's Equation in One Dimension	40
2.1.1	Green's Function	42
2.1.2	Smoothness of the Solution	43
2.1.3	A Maximum Principle	44
2.2	A Finite Difference Approximation	45
2.2.1	Taylor Series	46
2.2.2	A System of Algebraic Equations	47
2.2.3	Gaussian Elimination for Tridiagonal Linear Systems	50
2.2.4	Diagonal Dominant Matrices	53

2.2.5	Positive Definite Matrices	55
2.3	Continuous and Discrete Solutions	57
2.3.1	Difference and Differential Equations	57
2.3.2	Symmetry	58
2.3.3	Uniqueness	61
2.3.4	A Maximum Principle for the Discrete Problem . . .	61
2.3.5	Convergence of the Discrete Solutions	63
2.4	Eigenvalue Problems	65
2.4.1	The Continuous Eigenvalue Problem	65
2.4.2	The Discrete Eigenvalue Problem	68
2.5	Exercises	72
2.6	Projects	82
3	The Heat Equation	87
3.1	A Brief Overview	88
3.2	Separation of Variables	90
3.3	The Principle of Superposition	92
3.4	Fourier Coefficients	95
3.5	Other Boundary Conditions	97
3.6	The Neumann Problem	98
3.6.1	The Eigenvalue Problem	99
3.6.2	Particular Solutions	100
3.6.3	A Formal Solution	101
3.7	Energy Arguments	102
3.8	Differentiation of Integrals	106
3.9	Exercises	108
3.10	Projects	113
4	Finite Difference Schemes for the Heat Equation	117
4.1	An Explicit Scheme	119
4.2	Fourier Analysis of the Numerical Solution	122
4.2.1	Particular Solutions	123
4.2.2	Comparison of the Analytical and Discrete Solution . . .	127
4.2.3	Stability Considerations	129
4.2.4	The Accuracy of the Approximation	130
4.2.5	Summary of the Comparison	131
4.3	Von Neumann's Stability Analysis	132
4.3.1	Particular Solutions: Continuous and Discrete	133
4.3.2	Examples	134
4.3.3	A Nonlinear Problem	137
4.4	An Implicit Scheme	140
4.4.1	Stability Analysis	143
4.5	Numerical Stability by Energy Arguments	145
4.6	Exercises	148

5	The Wave Equation	159
5.1	Separation of Variables	160
5.2	Uniqueness and Energy Arguments	163
5.3	A Finite Difference Approximation	165
5.3.1	Stability Analysis	168
5.4	Exercises	170
6	Maximum Principles	175
6.1	A Two-Point Boundary Value Problem	175
6.2	The Linear Heat Equation	178
6.2.1	The Continuous Case	180
6.2.2	Uniqueness and Stability	183
6.2.3	The Explicit Finite Difference Scheme	184
6.2.4	The Implicit Finite Difference Scheme	186
6.3	The Nonlinear Heat Equation	188
6.3.1	The Continuous Case	189
6.3.2	An Explicit Finite Difference Scheme	190
6.4	Harmonic Functions	191
6.4.1	Maximum Principles for Harmonic Functions	193
6.5	Discrete Harmonic Functions	195
6.6	Exercises	201
7	Poisson's Equation in Two Space Dimensions	209
7.1	Rectangular Domains	209
7.2	Polar Coordinates	212
7.2.1	The Disc	213
7.2.2	A Wedge	216
7.2.3	A Corner Singularity	217
7.3	Applications of the Divergence Theorem	218
7.4	The Mean Value Property for Harmonic Functions	222
7.5	A Finite Difference Approximation	225
7.5.1	The Five-Point Stencil	225
7.5.2	An Error Estimate	228
7.6	Gaussian Elimination for General Systems	230
7.6.1	Upper Triangular Systems	230
7.6.2	General Systems	231
7.6.3	Banded Systems	234
7.6.4	Positive Definite Systems	236
7.7	Exercises	237
8	Orthogonality and General Fourier Series	245
8.1	The Full Fourier Series	246
8.1.1	Even and Odd Functions	249
8.1.2	Differentiation of Fourier Series	252
8.1.3	The Complex Form	255

8.1.4	Changing the Scale	256
8.2	Boundary Value Problems and Orthogonal Functions	257
8.2.1	Other Boundary Conditions	257
8.2.2	Sturm-Liouville Problems	261
8.3	The Mean Square Distance	264
8.4	General Fourier Series	267
8.5	A Poincaré Inequality	273
8.6	Exercises	276
9	Convergence of Fourier Series	285
9.1	Different Notions of Convergence	285
9.2	Pointwise Convergence	290
9.3	Uniform Convergence	296
9.4	Mean Square Convergence	300
9.5	Smoothness and Decay of Fourier Coefficients	302
9.6	Exercises	307
10	The Heat Equation Revisited	313
10.1	Compatibility Conditions	314
10.2	Fourier's Method: A Mathematical Justification	319
10.2.1	The Smoothing Property	319
10.2.2	The Differential Equation	321
10.2.3	The Initial Condition	323
10.2.4	Smooth and Compatible Initial Functions	325
10.3	Convergence of Finite Difference Solutions	327
10.4	Exercises	331
11	Reaction-Diffusion Equations	337
11.1	The Logistic Model of Population Growth	337
11.1.1	A Numerical Method for the Logistic Model	339
11.2	Fisher's Equation	340
11.3	A Finite Difference Scheme for Fisher's Equation	342
11.4	An Invariant Region	343
11.5	The Asymptotic Solution	346
11.6	Energy Arguments	349
11.6.1	An Invariant Region	350
11.6.2	Convergence Towards Equilibrium	351
11.6.3	Decay of Derivatives	352
11.7	Blowup of Solutions	354
11.8	Exercises	357
11.9	Projects	360
12	Applications of the Fourier Transform	365
12.1	The Fourier Transform	366
12.2	Properties of the Fourier Transform	368

12.3 The Inversion Formula	372
12.4 The Convolution	375
12.5 Partial Differential Equations	377
12.5.1 The Heat Equation	377
12.5.2 Laplace's Equation in a Half-Plane	380
12.6 Exercises	382
References	385
Index	389